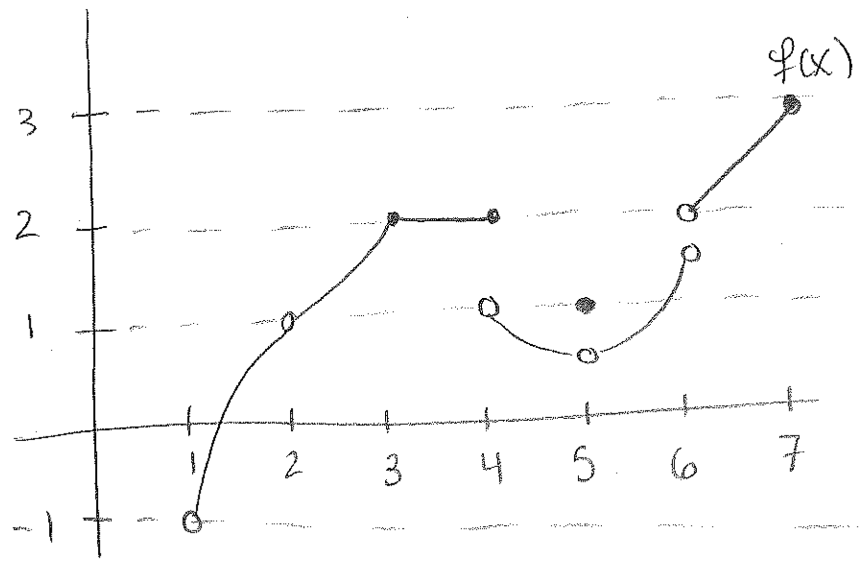
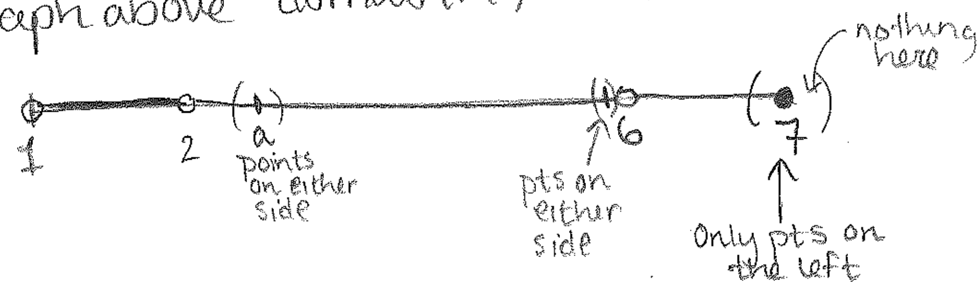
Jan. 15, 2014



Domain: $(1,2) \cup (2,6) \cup (6,7]$

Recall: An <u>interior pt.</u> is a point in the domain that has points in the domain on either side of it. — there don't have to be a lot of points, just an tiny bit is enough

in graph above domain: $(1,2) \cup (2,6) \cup (6,7]$



<u>Continuity Checklist:</u>

- If $x=a$ is an interior pt, then f is cont. at a if $\lim_{x \to a} f(x) = f(a)$

We know the following are cont. <u>on their domain:</u>

- polynomials
- rational functions
- exponentials (a^x, e^x)
- absolute values
- trig func
- inverses of all of these

Ex: Is $f(x) = \begin{cases} x^2 & x < 1 \\ x^3+2 & 1 \le x \end{cases}$ cont. everywhere?

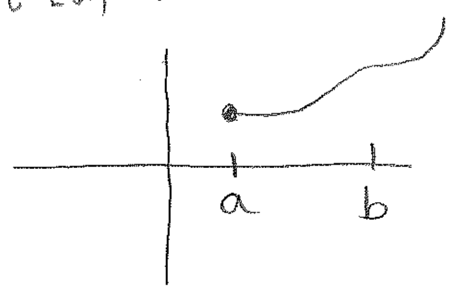
Know x^2, x^3+2 are continuous. So, just need to look at $x=1$ (notice this is an interior pt).

$$\underbrace{\lim_{x\to 1} f(x) = ?}_{\text{DNE}} \quad \left\{ \begin{array}{l} \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} x^2 = 1 \\ \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} x^3+2 = 3 \end{array} \right.$$

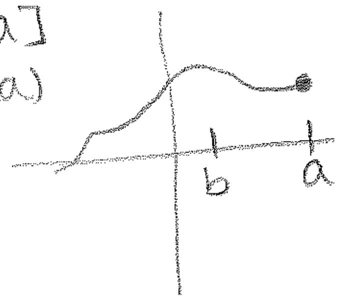
<u>Not continuous at $x=1$</u>

Left/Right Continuous

$f(x)$ is: • right cont. at $x=a$ if defined on $[a,b)$ and $\lim\limits_{x\to a^+} f(x) = f(a)$



• left cont at $x=a$ if defined on $(b,a]$ and $\lim\limits_{x\to a^-} f(x) = f(a)$



* these are weaker definitions of continuity, because you only look at the points on one side *

We will use this type of continuity for endpts.

(In example graph: $x = 7, 5, 4$ are not right cont
$x = 5$ is not left cont)

Checklist for continuity when $x=a$ is an endpt:

☐ 1) $x=a$ is an endpoint

☐ 2) either $\lim\limits_{x\to a^+} f(x)$ OR $\lim\limits_{x\to a^-} f(x)$ exists

☐ 3) $f(a) = \lim\limits_{x\to a^+} f(x)$ OR $f(a) = \lim\limits_{x\to a^-} f(x)$

(which ever one was defined)

Fixing & Filling

moves a point

- Removable Discontinuity

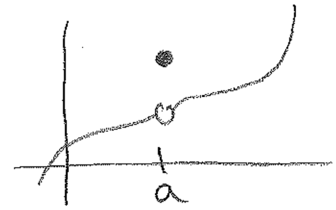
Let $x=a$ be a pt of discontinuity:
(i.e. not cont.)

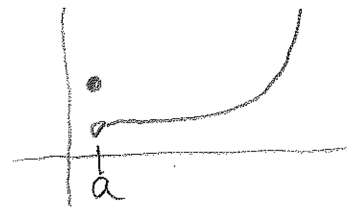
$x=a$ is a removable discont.
If $\lim_{x\to a} f(x)=L$ exists

(for $x=a$ endpt, $\lim_{x\to a^+} f(x)=L$ OR $\lim_{x\to a^-} f(x)=L$ exists)

Can define a cont. "fix"

$$\tilde{f}(x)=\begin{cases} f(x) & x\neq a \\ L & x=a \end{cases}$$





plugs in a point

- Continuous Extension

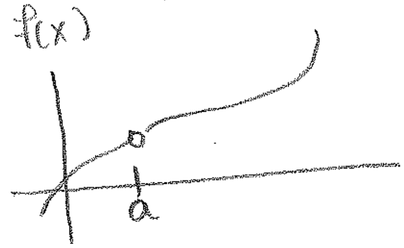
let $x=a$ be a "hole" in the domain of $f(x)$

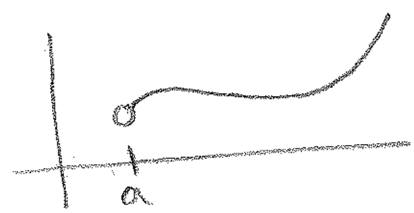
If $\lim_{x\to a} f(x)=L$ exists

(for $x=a$ endpt, want $\lim_{x\to a^-} f(x)=L$ OR $\lim_{x\to a^+} f(x)=L$ to exist)

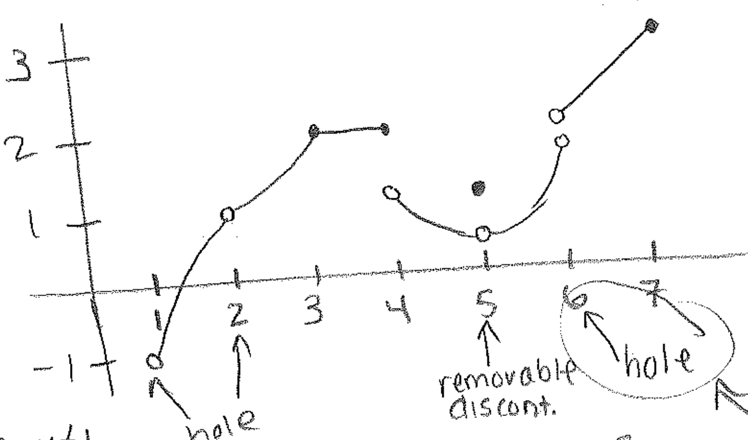
the we can define a cont. extension:

$$\bar{f}(x)=\begin{cases} f(x) & x\neq a \\ L & x=a \end{cases}$$





Example:



$$\bar{f}_1(x)=\begin{cases} f(x) & x\neq 1 \\ -1 & x=1 \end{cases}$$

$$\bar{f}_2(x)=\begin{cases} f(x) & x\neq 2 \\ 1 & x=2 \end{cases}$$

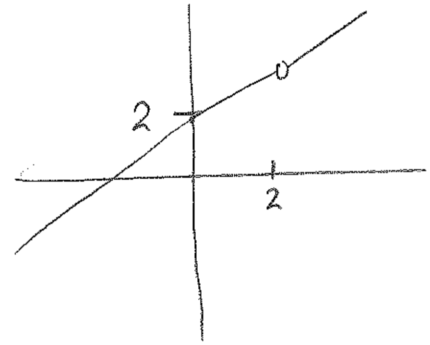
$$\underbrace{\tilde{f}(x)=\begin{cases} f(x) & x\neq 5 \\ 1/2 & x=5 \end{cases}}_{\text{fixes removable discontinuity}}$$

we can't fix this hole b/c $\lim_{x\to 6} f(x)=\text{DNE}$
So no linear ext.

$\underline{Ex}$: $f(x) = \underbrace{\dfrac{x^2-4}{x-2}}_{\text{look like}}$ D: $(-\infty, 2) \cup (2, \infty)$

Cont. extension? to fill in $x=2$

$y = x+2$ w/ a hole at $x=2$



$$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} x+2 = 4$$

$$\bar{f}(x) = \begin{cases} f(x) & x \neq 2 \\ 4 & x = 2 \end{cases} = x+2$$

$\underline{Ex}$: $f(x) = \begin{cases} \sin x & x \neq \pi/3 \\ 0 & x = \pi/3 \end{cases}$ continuous? check at $x = \pi/3$

$$\lim_{x \to \pi/3} f(x) = \lim_{x \to \pi/3} \sin x = \sin(\pi/3) = \sqrt{3}/2 \neq f(\pi/3) = 0$$

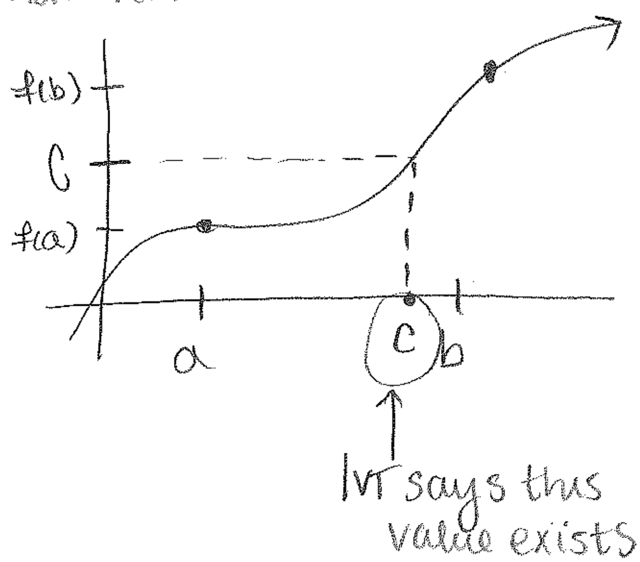
Removable discont. at $\pi/3$

$\tilde{f}(x) = \sin x$

An application of Continuity

Intermediate Value Thm:

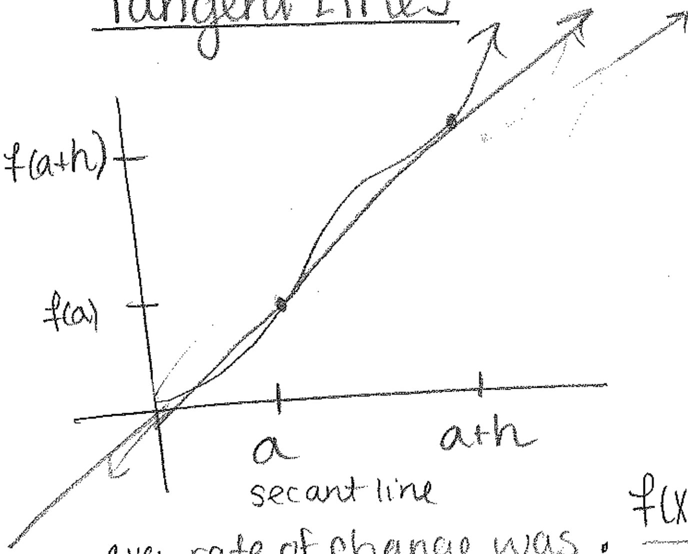
- f cont. on $[a, b]$
- Let C be a real # such that $f(a) < C < f(b)$ or $f(b) < C < f(a)$
- then, there exists c s.t. $f(c) = C$



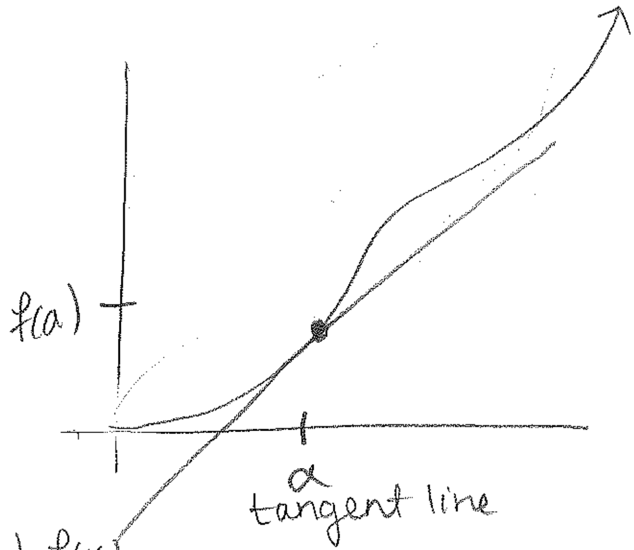
$\underline{Ex}$: Show $x^5 - 3x + 1 = 0$ has a sol'n in interval $[0, 1]$

- consider $f(x) = x^5 - 3x + 1$
- Notice $f(0) = 0 - 0 + 1 = 1$
  $f(1) = 1 - 3 + 1 = -1$
- If $C = 0$, then $f(1) < C < f(0)$
  $-1 < 0 < 1$
- by IVT, $\exists$ c s.t. $f(c) = 0$.
  So there is at least one root!

Tangent Lines



secant line



tangent line

avg. rate of change was slope of secant line : $\dfrac{f(x+h)-f(x)}{h}$

instantaneous rate of change is that slope as $h \to 0$: $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

this is the slope of the tangent line to $f(x)$ at $x=a$

Finding the Equation of a tangent line

Example: $f(x)=x^2$
Find equation of tangent line at $x=2$:

Slope: $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{f(2+h)-f(2)}{h}$

$= \lim\limits_{h\to 0} \dfrac{(2+h)^2-(2)^2}{h}$

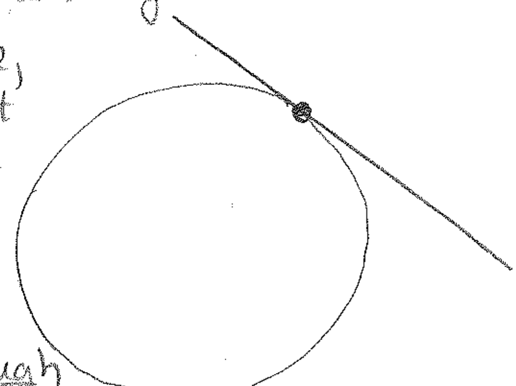
$= \lim\limits_{h\to 0} \dfrac{4+4h+h^2-4}{h}$

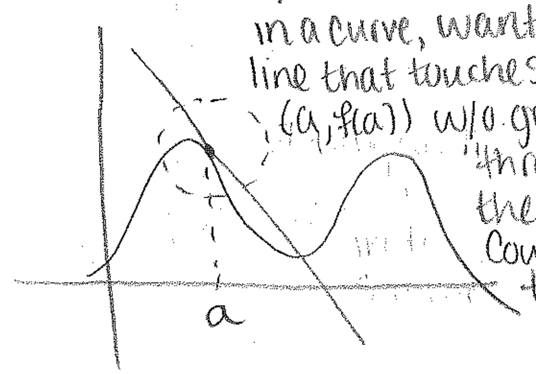
$= \lim\limits_{h\to 0} \dfrac{4h+h^2}{h} = \lim\limits_{h\to 0} 4+h = 4$

So slope $=4$. Use point-slope form for a line:
tangent line goes thru $(2,4)$ so $\boxed{y-4=4(x-2)}$

Side Note:
What are tangent lines?

In a circle, lines that touch the circle at precisely one pt, does not cross through the circle at the pt





in a curve, want the line that touches $(a, f(a))$ w/o going "through" the curve. Could go through the curve later.

Equation of Tangent Line

Have: $y=f(x)$

want tangent line at $x=a$

1) slope $=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=m$] hard part

2) equation: $y-f(a)=m(x-a)$

Need to be good at taking limits!

Ex: tangent line of $y=\sqrt{x}$ at $x=1$

1) slope $=\lim_{h\to 0}\frac{\sqrt{1+h}-\sqrt{1}}{h}$ TRICK: multiply top and bottom by conjugate of top.

$$=\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}\cdot\frac{(\sqrt{1+h}+1)}{(\sqrt{1+h}+1)}$$ difference of squares

$$=\lim_{h\to 0}\frac{(1+h)-(1)}{h(\sqrt{1+h}+1)}$$

$$=\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sqrt{1+h}+1)}=\lim_{h\to 0}\frac{1}{\sqrt{1+h}+1}=\frac{1}{\sqrt{1}+1}=\frac{1}{2}=m$$

2) Equation: tangent line goes through $(1,\sqrt{1})=(1,1)$

so $y-1=\frac{1}{2}(x-1)$

Ex: tangent line of $y=\frac{1}{1-2x}$ at $x=1$

$$\text{Slope: }\lim_{h\to 0}\frac{\frac{1}{1-2(1+h)}-\frac{1}{1-2(1)}}{h}=\lim_{h\to 0}\frac{\frac{1}{-1-2h}+1}{h}=\lim_{h\to 0}\frac{\left(\frac{1-1-2h}{-1-2h}\right)}{h}=\lim_{h\to 0}\frac{-2h}{h(-1-2h)}$$

$$=\lim_{h\to 0}\frac{-2}{-1-2h}=2$$

Equation: Goes thru $(1,-1)$

$y+1=2(x-1)$